From the secondary Steenrod algebra to $M\xi$

Algebraic Topology in memory of Hans-Joachim Baues

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The secondary Steenrod algebra $\mathcal{A}^{(2)}$ of Baues

Let $A =$ Steenrod algebra at a prime p.

Theorem (Baues (2006), Baues and Jibladze (2004))

There exists a "secondary Steenrod algebra" $\mathcal{A}^{(2)}$

$$\Sigma A \overset{\iota}{\rightarrowtail} \underset{B_0\text{-bimodule}}{B_1} \xrightarrow{\partial} \underset{\text{algebra}}{B_0} \twoheadrightarrow A$$

This computes 3-fold Massey products $\approx$ the d_2 differential in the ASS.

Given $a \cdot b = 0$, $b \cdot c = 0$ in A:

- first lift to B_0: $\tilde{a} \cdot \tilde{b} = \partial r, \quad \tilde{b} \cdot \tilde{c} = \partial s$
- then $\langle a, b, c \rangle \ni \iota^{-1}(r \cdot \tilde{c} - \tilde{a} \cdot s) \in \Sigma A$.

Theorem (N. (2012): smaller & more explicit model of $\mathcal{A}^{(2)}$)

Model for secondary Steenrod algebra for $p = 2$:

$$\Sigma A \rightarrowtail D_1 \xrightarrow{\partial} D_0 \twoheadrightarrow A$$

$$D_0 = \mathbb{Z}/4\mathbb{Z}\{\mathrm{Sq}(R)\} \oplus \sum\nolimits_{0\leq k<l}^{\oplus} \mathbb{Z}/2\mathbb{Z}\{Y_{k,l}\mathrm{Sq}(R)\}$$

D_0 represents formal power series modulo 4 under composition

$$f(x) = \sum \xi_k x^{2^k} + \sum_{0\leq k<l} 2\xi_{k,l} x^{2^k+2^l}$$

$Y_{k,l}$ dual to $x^{2^k+2^l}$, relations $\mathrm{Sq}(R)Y_{k,l} = \sum_{i,j} Y_{k+i,l+j}\mathrm{Sq}(R-\Delta_i-\Delta_j)$

$$Y_{k,l} = \begin{cases} Y_{l,k} & (l<k), \\ 2\mathrm{Sq}(\Delta_{k+1}) & (l=k). \end{cases}$$

Theorem (N. (2012): smaller & more explicit model of $\mathcal{A}^{(2)}$ (ctd.))

$$D_1 = \Sigma A \oplus \mu_0 \Sigma A \oplus \sum\nolimits_{0 \le k < l}^{\oplus} \mathbb{Z}/2\mathbb{Z}\{U_{k,l}\mathrm{Sq}(R)\}$$

with $\partial\mu_0 = 2$, $\partial U_{k,l} = Y_{k,l}$, $\mathrm{Sq}(R)\mu_0 = \mu_0\mathrm{Sq}(R) + \mathrm{Sq}(R - \Delta_1)$,

$$\mathrm{Sq}(R)U_{k,l} = \sum_{i,j} U_{k+i,l+j}\mathrm{Sq}(R - \Delta_i - \Delta_j)$$

with

$$U_{k,l} = \begin{cases} U_{l,k} + \mathrm{Sq}(\Delta_k + \Delta_l) & (l < k), \\ \mu_0\mathrm{Sq}(\Delta_{k+1}) + \mathrm{Sq}(2\Delta_k) & (l = k). \end{cases}$$

The $U_{k,l}$ come from a formal power series $f_2(x, y)$ in 2 variables:

$$Y_{k,l} \leftrightarrow x^{2^k + 2^l} \qquad U_{k,l} \leftrightarrow x^{2^k} y^{2^l}$$

A second variable is required since $U_{k,l} \neq U_{l,k}$

What does (D_0, D_1) represent?

Naive idea:

$f_1(x)$ looks roughly like a strict isomorphism between p-typical formal group laws F, G (modulo I^2), so

$$
\begin{aligned}
D_0 &: f_1(x) \qquad & f_1(x +_F y) &= f_1(x) +_G f_1(y) \\
D_1 &: f_2(x, y) \qquad & f_2(x, y) &= ???
\end{aligned}
$$

This explains $f_1(x)$ but creates an impossible riddle for $f_2(x, y)$.

What does (D_0, D_1) represent?

Not-so-naive-but-crazy idea:

$f_1(x)$ is a "homomorphism up to homotopy" between two "formal groups up to homotopy" F and G. $f_2(x, y)$ is the homotopy.

$$\text{“}\, f_1(x +_F y) = f_1(x) +_G f_1(y) +_G \partial f_2(x, y)\,\text{”}$$

The ∂ is not to be taken literally. The suggestion is that (f_1, f_2) behave formally like a homomorphism up to homotopy.

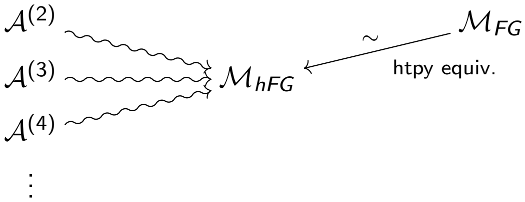
Higher order cohomology operations might require $f_r(x_1, \ldots, x_r)$ for $r > 2$, e.g.

$$\text{“}\, f_2(x +_F y, z) = f_2(x, y +_F z) +_G \partial f_3(x, y, z)\,\text{”}$$

The hypothetical $(f_1, f_2, f_3, \ldots)$ represents a homomorphism up to all higher coherence homotopies.

Fantasy: formal groups up to homotopy

Higher order Steenrod algebras $\mathcal{A}^{(2)}$, $\mathcal{A}^{(3)}$, ... can be related to a moduli space $\mathcal{M}_{hFG}$ of formal groups up to (coherent) homotopies. They can not be directly related to the moduli space $\mathcal{M}_{FG}$ of ordinary formal group laws.

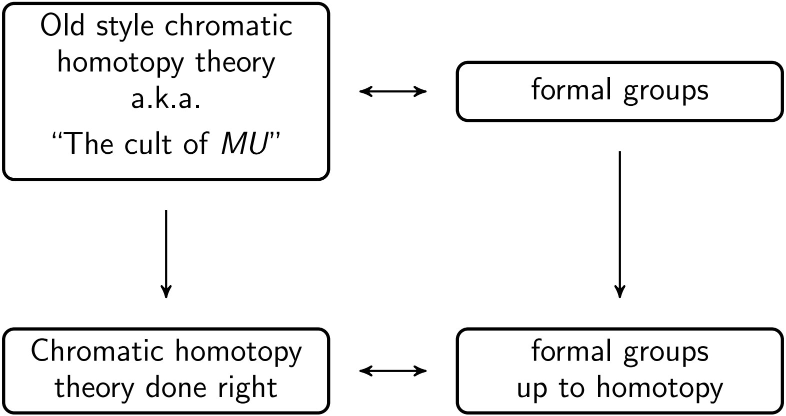


Being wobbly, free and homotopical is the natural, preferred state of a formal group law.

Every place in maths that uses formal groups should really use formal groups up to homotopy.

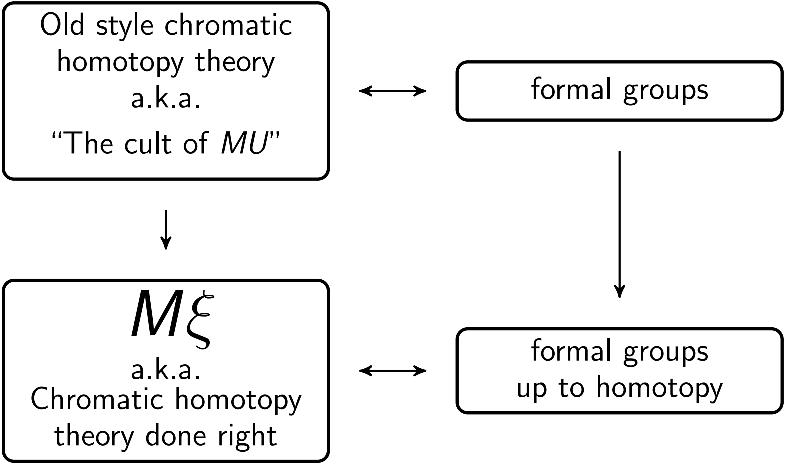
Fantasy: old vs. new style chromatic homotopy theory

Call for a revolution:



Fantasy: old vs. new style chromatic homotopy theory

The revolution has already taken place: join the <u>*Fellowship of $M\xi$*</u> !



Thm/Def (Baker and Richter (2008))

$$M\xi \stackrel{def}{=} \text{Thom}\left(\Omega\Sigma\mathbb{C}P^\infty \longrightarrow BU\right)$$

$M\xi$ is a complex-oriented, non-commutative A_∞ ring spectrum with a multiplicative map $M\xi \to MU$.

$$M\xi_{(p)} = \mathrm{BP} \otimes \textit{free } \mathrm{BP}_*\textit{-module}$$

$M\xi$ defines the same Adams spectral sequence as MU from E_2 onwards:

$$(E_r^{M\xi}, d_r) = (E_r^{MU}, d_r) \quad (r \geq 2)$$

E_1^{MU} can be understood as the nerve of the category/groupoid of formal groups and their isomorphisms. That groupoid defines $\mathcal{M}_{FG}$.

$E_1^{M\xi}$ has currently no such interpretation since *the $M\xi$-cooperations do not constitute a Hopf algebroid*.

Q: can $E_1^{M\xi}$ be understood as a quasi-category? It should define $\mathcal{M}_{hFG}$.

Hopf-algebroids in topology

The Hopf-algebroid for a ring spectrum E has

$$\mathrm{Ob} = \mathrm{Spec}\, E_*, \qquad \mathrm{Mor} = \mathrm{Spec}\, E_*E$$

with identity and composition

$$\mathrm{id} : \mathrm{Ob} \to \mathrm{Mor} \qquad \mathrm{comp} : \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor} \to \mathrm{Mor}$$

defined via ϵ, Φ below:

$$\begin{array}{ccccc} E_* & \underset{\epsilon}{\rightleftarrows} & E_*E & \rightleftarrows & \pi_*(E \wedge E \wedge E) \\ & & & & \Phi \uparrow \cong \\ & & & & E_*E \otimes_{E_*} E_*E \end{array}$$

Both ϵ and Φ are non-multiplicative if E is non-commutative, so neither id nor comp can be defined (N. (2002)).

So: what are formal groups up to homotopy?

From the hypothetical identification of $M\xi$ with a theory of formal groups up to homotopy we get a conjectural partial answer, resp. a different perspective on the question:

1. Over a commutative base ring R the homotopies play no role. A homotopical FG over R is the same as a classical FG over R.

2. Over a non-commutative base ring R there is no classical notion of a formal group over R. Conjecture: $M\xi$ can be used to define formal groups over non-commutative R, but the theory of such formal groups will be substantially homotopical.

Pointers & references

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